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A Theorem on the First Heteroclinic Tangency in Two-Dimensional Maps. Orientation-Preserving Cases

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Using the properties of the Jordan curve, the following theorem on the heteroclinic tangency in orientation-preserving two-dimensional maps is proved: Let T_μ: $\mathbf{R}^2 \to \mathbf{R}^2$ be a one-parameter family of C^1 diffeomorphisms and $J = \operatorname{Det} DT_\mu$ be such that $0 < J \leqslant 1$ or $1 \leqslant J < \infty$. Let $\mathbf{W}_u^n$ be the unstable manifold of a hyperbolic n-cycle and $\mathbf{W}_s^m$ the stable manifold of a hyperbolic m-cycle. Suppose that for $\mu < \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ have no common points, and that for $\mu > \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ have a transversal heteroclinic point. Then at $\mu = \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ are in the first asymptotic heteroclinic tangency except for the following three cases: (1) $n = m$; both cycles are without reflection. (2) $m = 2n$; the n- and m-cycles are with and without reflection, respectively; (3) $n = 2m$; the n- and m-cycles are without and with reflection, respectively.

KEY WORDS: Heteroclinic tangency; orientation-preserving map; hyperbolic point; unstable manifold; stable manifold; Jordan curve.

1. INTRODUCTION AND A THEOREM

The investigation of invariant manifolds is the shortest way to approach the heart of dynamical systems. Recently, there has been much work on invariant manifolds using various models.[1–13] The homoclinic or heteroclinic tangency between stable and unstable manifolds gives rise to various phenomena.[8,12] In ref. 13 we studied the mechanism of heteroclinic tangency by using a particular kind of two-dimensional map. It was shown that the existence of heteroclinic tangency in the usual sense

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is not obvious and evidently relies on the topology of stable and unstable manifolds and the properties of mapping functions. In this paper, we consider the first heteroclinic tangency between stable and unstable manifolds of any two hyperbolic points and show that the occurrence of heteroclinic tangency in the usual sense is rather exceptional. The definition of the first heteroclinic tangency is given below.

Let us consider a one-parameter family of two-dimensional C^1 diffeomorphisms

$$T_\mu: \quad \mathbf{R}^2 \to \mathbf{R}^2 \tag{1.1}$$

where μ is a bifurcation parameter. The Jacobian determinant $J = \text{Det}\, DT_\mu$ is assumed to satisfy

$$0 < J \leqslant 1 \qquad \text{or} \qquad 1 \leqslant J < \infty \tag{1.2}$$

Since the Jacobian is positive, the map T_μ is orientation preserving.[1,2]

Hereafter T_μ is abbreviated as T. Let us denote by T^n the n-folded iteration of the map ($n = \pm 1, \pm 2,...$). We call a periodic orbit of period n an n-cycle. Let q_1 and q_2 be hyperbolic points. A point P is called a heteroclinic point if $T^n P \to q_1 (n \to \infty)$ and $T^{-n} P \to q_2 (n \to \infty)$.[14]

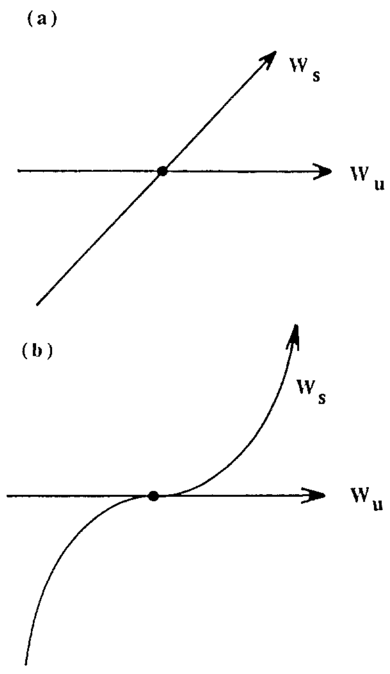

Fig. 1. Possible heteroclinic intersections. The intersection point is indicated by a solid circle.

Now let us change the parameter μ continuously and suppose that for $\mu < \mu_c$, the unstable manifold $\mathbf{W}_u^n$ from a point of a hyperbolic n-cycle and the stable manifold $\mathbf{W}_s^m$ from a point of a hyperbolic m-cycle have no common points, and that for $\mu > \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ have a transverse heteroclinic point (see Fig. 1a). We say that at $\mu = \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ are in the *first* heteroclinic tangency. Note that there cannot be a nontransversal heteroclinic point at $\mu = \mu_c$ since it cannot be removed by a small change of μ. As Newhouse[8] showed in the homoclinic case, there can be a lot of tangency points once stable and unstable manifolds intersect transversely. These tangencies are not the first ones.

Let us classify the first heteroclinic tangency. Let $[\mathbf{A}]$ denote the closure of a set $\mathbf{A}$ and $\partial\mathbf{A} = [\mathbf{A}] - \mathbf{A}$. If $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ are tangent to each other at some point but never intersect at any point, we say that they are in the *first direct* heteroclinic tangency. We say that $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ are in the *first asymptotic* heteroclinic tangency if they have no common points, whereas $[\mathbf{W}_u^n]$ and $[\mathbf{W}_s^m]$ have common points. Then the first heteroclinic tangency is divided into four classes.

(he-1) $\mathbf{W}_u^n \cap \mathbf{W}_s^m \neq \varnothing$.

(he-2) $\partial\mathbf{W}_u^n \cap \mathbf{W}_s^m \neq \varnothing$ and $\mathbf{W}_u^n \cap \partial\mathbf{W}_s^m = \mathbf{W}_u^n \cap \mathbf{W}_s^m = \varnothing$.

(he-3) $\mathbf{W}_u^n \cap \partial\mathbf{W}_s^m \neq \varnothing$ and $\partial\mathbf{W}_u^n \cap \mathbf{W}_s^m = \mathbf{W}_u^n \cap \mathbf{W}_s^m = \varnothing$.

(he-4) $\partial\mathbf{W}_u^n \cap \partial\mathbf{W}_s^m \neq \varnothing$ and $\partial\mathbf{W}_u^n \cap \mathbf{W}_s^m = \mathbf{W}_u^n \cap \partial\mathbf{W}_s^m = \mathbf{W}_u^n \cap \mathbf{W}_s^m = \varnothing$.

A similar situation for the *first homoclinic* tangency was considered by Mañè.[9] He called common points almost homoclinic points. But he did not consider the case (he-4).

The main purpose of this paper is to give the following theorem on the first heteroclinic tangency.

Theorem. Let T_μ: $\mathbf{R}^2 \to \mathbf{R}^2$ be a one-parameter family of C^1 diffeomorphisms and $J = \operatorname{Det} DT_\mu$ be such that $0 < J \leqslant 1$ or $1 \leqslant J < \infty$. Let $\mathbf{W}_u^n$ be the unstable manifold of a hyperbolic n-cycle and $\mathbf{W}_s^m$ the stable manifold of a hyperbolic m-cycle. Suppose that for $\mu < \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ have no common points, and that for $\mu > \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ have a transversal heteroclinic point. Then at $\mu = \mu_c$, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ are in the first asymptotic heteroclinic tangency except for the following three cases:

(1) $n = m$. Both cycles are without reflection.

(2) $m = 2n$. The n- and m-cycles are with and without reflection, respectively.

(3) $n = 2m$. The n- and m-cycles are without and with reflection, respectively.

In these three cases, $\mathbf{W}_u^n$ and $\mathbf{W}_s^m$ can be in the first direct heteroclinic tangency.

Let us call above three cases the exceptional cases.

Examples of asymptotic tangency are illustrated in ref. 13. We believe that the $\partial \mathbf{W}_u^n$ is closely related to the strange attractor in dissipative systems.[3] Detailed discussions are given elsewhere.[15]

The orientation preservation of the map and the Jordan curve theorem play essential roles in the proof. The latter is very powerful when we consider the topological behavior of invariant curves or study the coexistence of periodic points.[16,17] The properties of the Jordan curve are discussed in ref. 18. In Section 2, notation is introduced and the proof of the theorem is obtained.

2. PROOF OF THE THEOREM

2.1. Orientation-Preserving Map

First we review the concept of orientation preservation (see Chapter 1 in ref. 2). For a hyperbolic point with positive Jacobian, there are two cases of orientation preservation:

$$\text{Case-OP1:} \quad \lambda_1 > 1 > \lambda_2 > 0$$

$$\text{Case-OP2:} \quad \lambda_1 < -1 < \lambda_2 < 0$$

where $\lambda_1 \lambda_2 > 0$.

A hyperbolic point with case-OP1 (-OP2) is called a hyperbolic point without (with) reflection. When we draw a picture of the tangency between the unstable and stable manifolds, we take into account the difference of the two cases. In Fig. 2 an example is illustrated. When two manifolds are tangent at the point P as shown in Fig. 2a, they must be tangent at the point TP as shown in Fig. 2b (case-OP1) or Fig. 2c (case-OP2).

2.2. Notation on Stable and Unstable Manifolds and on Hyperbolic points

Here we introduce notation on manifolds and hyperbolic points.

$\mathbf{W}_u^n$(op1): The unstable manifold of a point of a hyperbolic n-cycle with OP1.

[3] This statement is our conjecture, not proved rigorously. Eckmann and Ruelle[20] state that the strange attractor is the union of unstable manifolds.

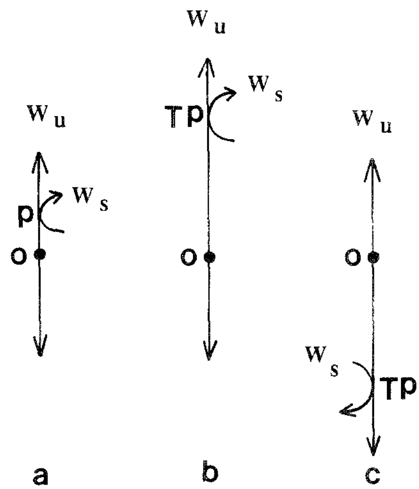

Fig. 2. Examples of the unstable ($\mathbf{W}_u$) and stable ($\mathbf{W}_s$) manifolds tangent to each other with the orientation preservation. (a) They are tangent at **P**. A hyperbolic fixed point is shown by **o**. (b) They are tangent at *TP* for the case OP1. (c) They are tangent at *TP* for the case OP2.

$\mathbf{W}_u^n$(op2): The unstable manifold of a point of a hyperbolic n-cycle with OP2.

$\mathbf{W}_s^n$(op1): The stable manifold of a point of a hyperbolic n-cycle with OP1.

$\mathbf{W}_s^n$(op2): The stable manifold of a point of a hyperbolic n-cycle with OP2.

$\mathbf{W}_u$: The abbreviated name of unstable manifolds.

$\mathbf{W}_s$: The abbreviated name of stable manifolds.

u_i: The hyperbolic points from which $\mathbf{W}_u^n$(op1 or op2) comes $(1 \leqslant i \leqslant n)$ and $u_i = T^{i-1}u_1$ and $u_{n+1} = u_1$.

s_i: The hyperbolic points to which $\mathbf{W}_s^n$(op1 or op2) tends $(1 \leqslant i \leqslant n)$ and $s_i = T^{i-1}s_1$ and $s_{n+1} = s_1$.

2.3. A Lemma on the First Direct Tangency

In this subsection, we give a lemma required in the proof of the theorem. Let us introduce an orientation on both stable and unstable manifolds as that to which points move upon iteration of T.

Lemma. The orientations of the stable and unstable manifolds coincide at the first direct homoclinic and heteroclinic tangency points.

Proof. (1) Case of the first direct homoclinic tangency.

Let us consider the orientation-preserving case. Assume the contrary, that the orientations of the stable and unstable manifolds do not coincide. Then the situation can be illustrated as in Fig. 3a, where q_1 is a point of a hyperbolic n-cycle, and the $\mathbf{W}_s$ and $\mathbf{W}_u$ are the stable and unstable manifolds. A Jordan curve Γ is constructed by the Arc q_1P on $\mathbf{W}_u$ and Arc Pq_1 on $\mathbf{W}_s$. The orientation of Γ is shown by arrows. The arrows **a** and **b** are on the different sides of Γ. On the other hand, the arc of $\mathbf{W}_u$ around P is mapped to the arc of $\mathbf{W}_u$ around T^nP by T^n. Then the arc of $\mathbf{W}_u$ connecting the arrows **a** and **b** must intersect the Arc Pq_1 of $\mathbf{W}_s$ since $\mathbf{W}_u$ cannot intersect itself. Thus, there exists at least one intersection point of $\mathbf{W}_s$ and $\mathbf{W}_u$, which is a contradiction.

For the orientation-reversing case, the same argument applies using $2n$ instead of n.

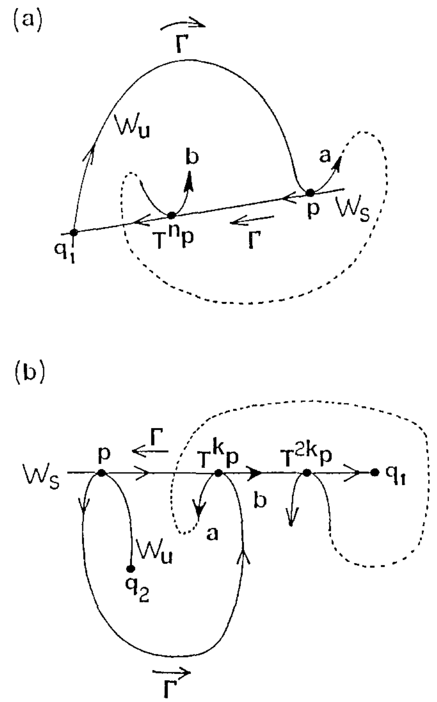

Fig. 3. (a) The first direct homoclinic tangency where q_1 is a point of a hyperbolic n-cycle (see text). (b) The first direct heteroclinic tangency where q_1, q_2, respectively, are points of hyperbolic m- and n-cycles and k is the least common multiple of n and m. The Jordan curve Γ is constructed by Arc PT^kP of $\mathbf{W}_u$ and Arc T^kPP of $\mathbf{W}_s$, and the orientation of Γ is shown by the arrows. The arrows **a** and **b** are on different sides of Γ. The arc connecting T^kP and $T^{2k}P$ must intersect Γ.

(2) Case of the first direct heteroclinic tangency.

The situation is illustrated in Fig. 3b. The proof is similar to the above. The orientation-reversing case can be treated using $2k$ instead of k.

Then the lemma is proved. ∎

2.4. Proof of the Theorem

First we give two remarks:

Remark 1. When the unstable manifold $\mathbf{W}_u^n(\mathrm{op1})$ and the stable manifold $\mathbf{W}_s^m(\mathrm{op1})$ are tangent at the point P, these manifolds are tangent at the sequential points $T^{\pm ik}P$, where $k = \mathrm{LCM}(n, m)$ is the least common multiple of n and m, and i is a positive integer. When $\mathbf{W}_u^n(\mathrm{op1})$ and $\mathbf{W}_s^m(\mathrm{op2})$ [$\mathbf{W}_u^n(\mathrm{op2})$ and $\mathbf{W}_s^m(\mathrm{op1})$] are tangent at P, two manifolds are tangent at the sequential points $T^{\pm ik}P$, where $k = \mathrm{LCM}(n, 2m)$ [$k = \mathrm{LCM}(2n, m)$]. When $\mathbf{W}_u^n(\mathrm{op2})$ and $\mathbf{W}_s^m(\mathrm{op2})$ is tangent at P, two manifolds are tangent at the sequential points $T^{\pm ik}P$, where $k = \mathrm{LCM}(2n, 2m)$. These are obvious consequence of the map T.

Remark 2. From the lemma given in Section 2.3, the orientations of stable and unstable manifolds coincide at P, and then the cases that the orientations of such manifolds are opposite each other at P are omitted in the proof.

We divide the proof into eight cases as shown in Table I according to

Table I. Situations of Heteroclinic Tangency for Orientation-*Preserving* Cases

		$\mathbf{W}_s^m$	
		OP1	OP2
$\mathbf{W}_u^n$	OP1	(1) $n<m$ (1′) $n=m$	(2) $n<m$ (2′) $n=m$
	OP2	(3) $n<m$ (3′) $n=m$	(4) $n<m$ (4′) $n=m$
		$\mathbf{W}_u^m$	
		OP1	OP2
$\mathbf{W}_s^n$	OP1	(5) $n<m$	(6) $n<m$
	OP2	(7) $n<m$	(8) $n<m$

the values of n and m, and also to the values of λ_1 and λ_2. In each case, we assume that stable and unstable manifolds are tangent at some point and have no common intersection points. We derive a contradiction under this assumption. If a contradiction is not derived, then the first direct heteroclinic tangency may occur for the corresponding case.

Case 1. Tangency between $\mathbf{W}_u^n(\mathrm{op1})$ and $\mathbf{W}_s^m(\mathrm{op1})$ for $m \geqslant n$. Let us first consider the case $m > n$. According to the two remarks and the orientation preservation of T, the situation is schematically illustrated in Fig. 4. The unstable manifold from u_1 and the stable manifold from s_1 are tangent at P. The unstable manifold is tangent to the stable manifold from s_{n+1} at T^nP, and it is also tangent to the stable manifold from s_1 at T^kP, where $k = \mathrm{LCM}(n, m)$. Note that the point $T^{n+k}P$ is both on the unstable manifold and on the stable manifold from s_{n+1}.

A Jordan curve Γ is constructed by two arcs: Arc PT^kP on the unstable manifold and Arc T^kPP on the stable manifold from s_1. The orientation on Γ is defined as that of the unstable manifold (shown by the dashed arrow). The arrows **a** and **b** are on different sides of Γ. The point $T^{n+k}P$ must be on the extension of the arrow **b**. The Jordan curve connecting the arow **b** and $T^{n+k}P$ (shown by the dotted curve) must intersect Γ. Since the unstable manifold cannot intersect itself, then Arc $\mathbf{b}T^{n+k}P$ must intersect Arc T^kPP on the manifold from s_1. Then a contradiction is derived for case 1 with $m > n$.

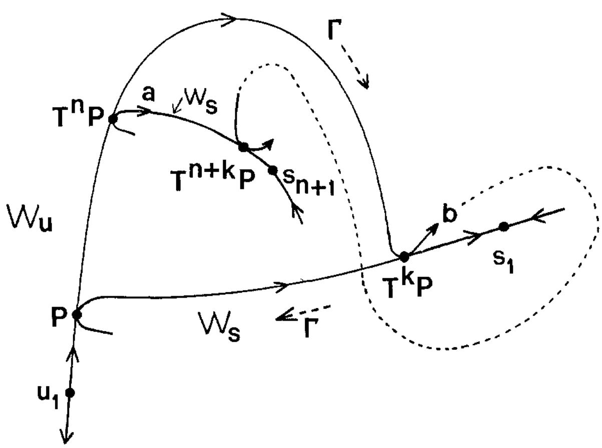

Fig. 4. The unstable manifold from a point of a hyperbolic n-cycle and the stable manifold from a point of a hyperbolic m-cycle. The dashed curve shows the orientation on the Jordan curve Γ. For case 1 with $m > n$, u_1 (resp. s_1) is a point of a hyperbolic n-cycle (resp. m-cycle) with OP1. For case 2 with $m > n$, u_1 (resp. s_1) is a point of a hyperbolic n-cycle (resp. m-cycle) with OP1 (resp. OP2).

Consider the cases with $n=m$, which is case 1 in the theorem. In this case, the point T^nP is equal to the point T^kP. We cannot derive a contradiction by way of Fig. 4.

We comment on Fig. 4. We have drawn it for the case in which the arrow **b** is on the left of Γ. The case in which the arrow **b** is on the right of Γ is easily constructed by visualizing Fig. 4 from the back side of the paper.

Case 2. Tangency between $\mathbf{W}_u^n(\mathrm{op1})$ and $\mathbf{W}_s^m(\mathrm{op2})$ for $m \geqslant n$. First consider the case with $m > n$. This case can be reduced to case 1 with $m > n$. In fact, a hyperbolic m-cycle with OP2 can be regarded as a hyperbolic $2m$-cycle with OP1, and hence Fig. 4 can be used with $2m$ instead of m.

Next consider the case with $n=m$. In this case, we cannot use Fig. 4 as it is, since $k=2n$. We use another situation as shown in Fig. 5. The stable manifold bound for the right side of s_1 is tangent at the point T^nP to the unstable manifold because $n=m$ and the point s_1 is one of hyperbolic points with OP2. Such a curve must intersect Γ constructed by two arcs: Arc $PT^{2n}P$ on the unstable manifold and Arc $T^{2n}PP$ on the stable manifold. The stable manifold cannot intersect itself and then Arc T^nPs_1 must intersect Arc $PT^{2n}P$ on the unstable manifold. A contradiction is derived for case 2.

Case 3. Tangency between $\mathbf{W}_u^n(\mathrm{op2})$ and $\mathbf{W}_s^m(\mathrm{op1})$ for $m \geqslant n$. First consider the case with $n \neq m$ and $m \neq 2n$. The situation is illustrated in Fig. 6. Here $k = \mathrm{LCM}(2n, m)$, and the index N is the minimum positive integer satisfying $N-1 = 2n(\mathrm{mod}\, m)$. The relations $2n < k$ and $N \neq 1$ are

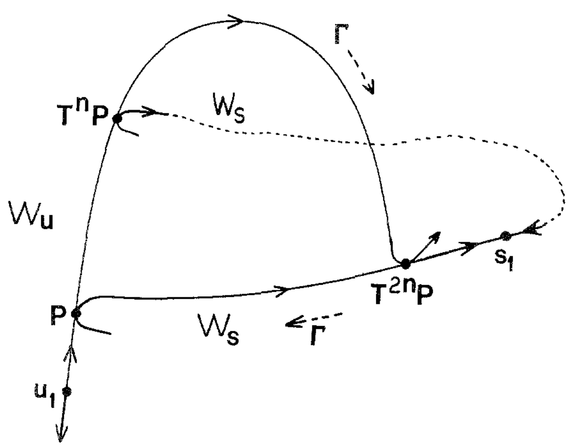

Fig. 5. The situation of case 2 with $m=n$.

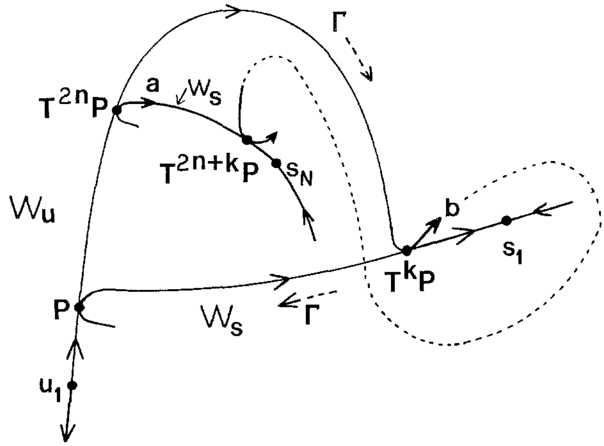

Fig. 6. For case 3 with $m>n$, u_1 (resp. s_1) is a point of a hyperbolic n-cycle (resp. m-cycle) with OP2 (resp. OP1). For case 4 with $m>n$ and $m\neq 2n$, u_1 (resp. s_1) is a point of a hyperbolic n-cycle (resp. m-cycle) with OP2.

satisfied. In Fig. 6, Arc PT^kP on the unstable manifold and Arc T^kPP on the stable manifold form a Jordan curve. The arrows **a** and **b** are on different sides of Γ. The point $T^{2n+k}P$ must be on the prolongation of the arrow **b**. The curve connecting the arrow **b** and $T^{2n+k}P$ (shown by the dotted curve) must intersect Γ.

Next we consider the case with $n=m$. In this case, $k=2n$. This situation is shown in Fig. 7. The branch of the unstable manifold bound for the

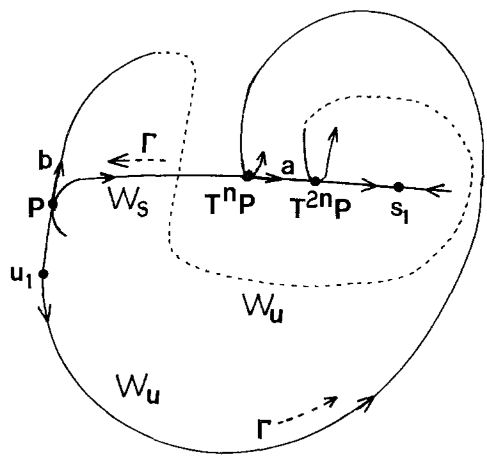

Fig. 7. The situation of case 3 with $n=m$.

bottom must be tangent at the point T^nP to the stable manifold bound for s_1. On the other hand, the branch of the unstable manifold bound for the top must be tangent to the stable manifold from s_1 at the point $T^{2n}P$, and the point $T^{2n}P$ is between T^nP and s_1 on the stable manifold. Then the arc of unstable manifold connecting P and $T^{2n}P$ necessarily intersects Γ.

Consider the cases with $m=2n$. In this case, $k=2n$ and then the point $T^kP=T^{2n}P$ in Fig. 6. The point T^nP in Fig. 7 is not on the stable manifold from s_1 when $m=2n$. Then we cannot derive a contradiction using Fig. 7. This is the exceptional case 2 in the theorem. Using the $2n$-folded map, the exceptional case 2 with $m=2n$ becomes the exceptional case 1 with $m=n$.

Case 4. Tangency between $\mathbf{W}_u^n(\text{op2})$ and $\mathbf{W}_s^m(\text{op2})$ for $m \geqslant n$. Using new notations $n'=2n$ and $m'=2m$, it seems that this case is formally the same as case 1. But we use the same approach as in the previous cases to derive a contradiction because there exists a particular case with $n=m$. The situation of case 4 with $n=m$ is different from that of case 1. This will be discussed later.

First we consider the case with $m>n$ and $m \neq 2n$. Figure 5 applies, where u_1 is a point of a hyperbolic n-cycle with OP2 and s_1 is a point of a hyperbolic m-cycle with OP2. Let $k=\text{LCM}(2n, 2m)$ and the index N be the minimum positive integer satisfying $N-1=2n(\text{mod}\, m)$. Then they satisfy two relations: $2n<k$ and $N \neq 1$.

By a similar reasoning as in case 3, we can show that a curve connecting the arrow $\mathbf{b}$ and $T^{n+k}P$ (shown by the dotted curve) must intersect Γ.

Next consider the cases with $m=2n$. In this case, $k=4n$ and $N=1$. The untable manifold bound for the top is tangent to the stable manifold from s_1 bound for the left side at $T^{4n}P$ (see Fig. 8). On the other hand, the

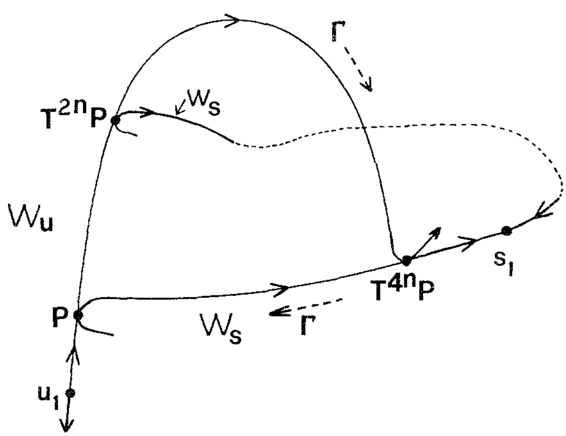

Fig. 8. The situation of case 4 with $m=2n$.

stable manifold bound for the right side must be tangent to the unstable manifold bound for the top at $T^{2n}P$. Then the curve connecting such two intersects Γ.

Next consider the case with $n = m$. Figure 6 does not apply since the point $T^{2n}P$ coincides with T^kP and N is equal to one. We apply the area-contracting, -expanding, or -preserving property of the maps. In Fig. 9 we show a typical example. The points $T^{-3n}P$, $T^{-n}P$, T^nP,... are on both the stable manifold bound for the right and the unstable manifold bound for the bottom. The points $T^{-4n}P$, $T^{-2n}P$, P, $T^{2n}P$,... are on both the stable manifold bound for the left and the unstable manifold bound for the top. A Jordan curve is constructed by two arcs: Arc Pu_1T^np on the unstable manifold and Arc Ps_1T^nP on the stable manifold. Let **D** be the area of the region surrounded by the Jordan curve. If $0 < J < 1$, the area A of the hatched region increases as A/J^{2n} by T^{-2n}. Therefore, the area exceeds **D** by a finite backward iteration. For $J = 1$, the summed area of a finite backward iteration of the hatched region exceeds **D**. If $1 < J < \infty$, the area B of the dotted region exceeds **D** by a finite forward iteration. In any case, a contradiction is derived.

As a result, a contradiction is derived for case 4.

Case 5. Tangency between $\mathbf{W}_s^n(\mathrm{op1})$ and $\mathbf{W}_u^m(\mathrm{op1})$ for $m > n$.

Case 6. Tangency between $\mathbf{W}_s^n(\mathrm{op1})$ and $\mathbf{W}_u^m(\mathrm{op2})$ for $m > n$.

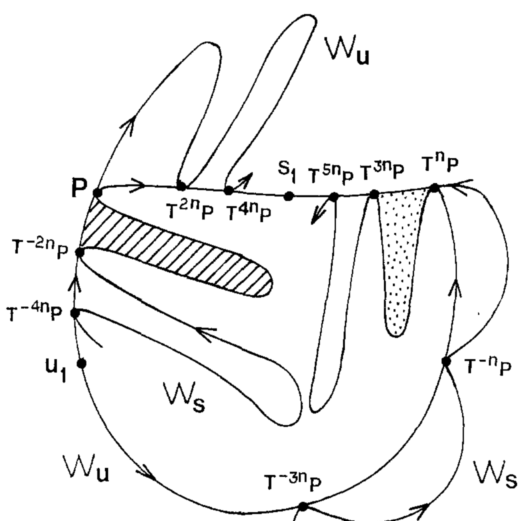

Fig. 9. The situation of case 4 with $m = n$, where A and B are the areas of the hatched and dotted regions, respectively. See text.

Case 7. Tangency between $\mathbf{W}_s^n(\mathrm{op2})$ and $\mathbf{W}_u^m(\mathrm{op1})$ for $m > n$.

These cases reduce, respectively, to cases 1, 2, and 3 using T^{-1} instead of T.

Case 8. Tangency between $\mathbf{W}_s^n(\mathrm{op2})$ and $\mathbf{W}_u^m(\mathrm{op2})$ for $m > n$.

This case reduces to the case 5 with $2n$ and $2m$ instead of n and m.

2.5. Miscellaneous Comments on the Theorem

1. Can we change the condition "$0 < J \leqslant 1$ or $1 \leqslant J < \infty$" to "$0 < J < \infty$" in the theorem? The answer is no, because a condition on the monotone increase and decrease of the area under the iteration of T is used in case 4 with $n = m$.

2. The occurrence of first direct heteroclinic tangency can be excluded if there is an additional symmetry in the maps, say the symmetry of mapping functions. In general, the occurrence cannot be excluded only by the Jordan curve theorem. The theorem makes clear the situations in which the first direct heteroclinic tangency occurs.

3. The theorem on the heteroclinic tangency in the orientation-*reversing* maps will be reported elsewhere.[19]

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